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ALL-POLE MODELS OF AUDITORY FILTERING

R.F. LYON
Apple Computer, Inc., One Infinite Loop
Cupertino, CA 94022 USA
lyon@apple.com

The all-pole gammatone filter (APGF), which we derive by discarding the zeros from the popular gammatone filter (GTF), and the related all-pole filter cascade (APFC) provide models of auditory filtering with simple parameterization and useful properties. Compared to the GTF, these all-pole models offer fewer parameters, a more controlled behavior of the tuning-curve tail, and an easier way to model level-dependent gain, bandwidth, asymmetry, and center-frequency (CF) shift. The order-N APGF is the Nth power of a filter with one complex-conjugate pair of poles; the GTF has this same set of poles, but, in addition, has spurious zeros on the real axis that complicate its description and behavior. Since the APGF is the Nth power of a simple pole-pair filter, many of its properties—such as gain, bandwidth, delay, and dispersion—are easy to compute as analytic functions of the parameters, which may be tied to sound level or to output power level. Good approximate and empirical results extend these functions to the APFC, which is an efficient tapped filter-cascade structure that has a mathematical link to an underlying traveling-wave structure via the Liouville–Green (or WKB) approximation. Fixing the all-pole filter's tail gain, rather than fixing the peak gain, as we vary parameters with sound level, allows us to construct more realistic models of how the auditory system behaves across levels, emphasizing approximate linearity at low frequencies; variation in the damping parameter of the poles can then be interpreted in terms of automatic gain control near CF. The APGF and APFC represent two steps from the popular GTF toward more flexible, realistic, physically grounded, and efficiently implemented models of auditory filtering.

1 Models of Auditory Filtering

Models of auditory filtering are useful tools for understanding, describing, and emulating experiments in auditory mechanics, auditory physiology, and auditory psychophysics. Linear time-invariant lumped-parameter models (rational transfer functions) form a useful class of filters on which to build, even when we are modeling nonlinear and time-varying properties of the auditory system. The gammatone filter (GTF) is a simple and easily parameterized member of this class that has been used in models of mechanics, psychophysics, and physiology. By contrast, the rounded exponential (roex) filters that are often used in such models are not members of this class, since there is no known phase response that makes the transfer function into a rational function (ratio of polynomials) in the Laplace transform domain; roex filters therefore cannot be implemented as circuits.

In this paper, we consider a further restriction to the class of easily parameterized *all-pole* filter models. We show that the GTF has zeros (i.e., the numerator in its rational transfer function is not a scalar, but rather is a polynomial,

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whose roots are the zero locations), and that by removing them we arrive at the all-pole gammatone filter (APGF), which has improved properties for auditory modeling.

The GTF and APGF of order *N* have *N* coincident poles at a complex-conjugate pair of locations in the Laplace transform's *s* plane. Relaxing this constraint of coincident poles, we consider filters whose poles are in differing locations, and particularly filterbanks made up of cascades of two-pole filters with progressively varying pole locations—the all-pole filter cascade (APFC). These filters are closely related to wave-propagation models of the underlying cochlear mechanics, and form a bridge between cochlear-mechanics models and the auditory-filter models used in psychophysics and physiology.

Filters with zeros are still of interest, as well, but zeros need to be considered carefully as targeted modifications of all-pole filters, rather than to be accepted as ill-behaved interference effects—as they are in the GTF.

2 Gammatone Filters With and Without Zeros

The GTF and the APGF are closely related auditory-filter models, sharing a complex conjugate pair of order-*N* poles. The GTF also has *N* zeros on the real axis in the *s* plane, inherited from its defining impulse response via the Laplace transform. We define the APGF by discarding those zeros, making it an all-pole filter.

We describe the APGF in terms of its Laplace transform $H(s)$. For conciseness, let the *s*-plane pole positions be given by the complex number p and its conjugate p^*. Then the APGF is given by:

$$H_{AP}(s) = \frac{K}{[(s-p)(s-p^*)]^N} \tag{1}$$

where K is a constant that we will normally adjusted to give unity gain at DC: $H(0)=1$. In terms of the Cartesian parameterization of complex pole position $p=-b+i\omega_r$ (as typically used with the GTF), the APGF becomes

$$H_{AP}(s) = \frac{b^2+\omega_r^2}{[(s+b)^2+\omega_r^2]^N} \tag{2}$$

The GTF impulse response $g(t)=t^{N-1}\exp(-bt)\cos(\omega_r t+\phi)$—a gamma distribution times a tone—transforms to a rational transfer function with the same poles as the APGF, but with a complicated numerator with N real roots:

$$H_{GT}(s) = \frac{e^{i\phi}[s+b+i\omega_r]^N + e^{-i\phi}[s+b-i\omega_r]^N}{[(s+b)^2+\omega_r^2]^N} \tag{3}$$

The roots of the numerator—the zeros of the GTF—fall on the real axis in the *s* plane, at locations that depend on all the parameters. By varying the phase or

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bandwidth parameters, we can make a zero fall exactly at $s = 0$, so the tail gain can go to zero for particular parameter values.

The differentiated APGF (DAPGF) also has the same poles, and one zero at DC (a numerator factor of s, or differentiator); thus, it is closely related to both the APGF and the GTF:

$$H_D(s) = \frac{Ks}{[(s+b)^2 + \omega_r^2]^N} \tag{4}$$

On a continuum between the APGF and the DAPGF is the one-zero gammatone filter (OZGF), in which the generalized numerator $K(s + \omega_z)$ has *one zero* anywhere on the real axis. We can convert a GTF of any phase to an OZGF by discarding all but one of its zeros.

The GTF and DAPGF, in the form of Laplace transform transfer functions of order $N = 3$, were first used by Flanagan to model basilar membrane motion.[1] Slaney analyzed the GTF and APGF Laplace transforms, including the zeros of the GTF.[2]

Figure 1 shows typical GTF, APGF, and DAPGF transfer functions; Figure 2 shows the corresponding impulse responses. Note that the GTF tail gain depends on the phase parameter (in a complicated interaction with the bandwidth and order parameters), whereas the tails of the APGF and DAPGF are fixed, independent of any other parameters.

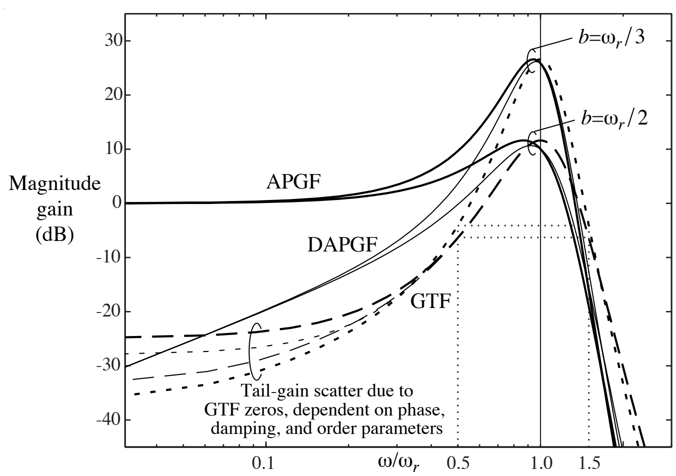

Figure 1: Comparison of GTF, APGF, and DAPGF transfer functions ($N = 6$) for two different values of the real part of the pole location. Dotted lines show the symmetry of the GTF at $f = 1 \pm 0.5$. Each GTF is shown with two phases (0 and $\pi/2$, heavy and light dashed lines, respectively). Notice that the ordering of gain near the peak is not maintained in the tail. The variation in the GTF tail is not accounted for by the usual phase-independent symmetric GTF approximation.

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Since the APGF is just the *N*th power of the well-studied two-pole filter, its properties—e. g., bandwidth, center frequency (CF), delay, and dispersion—are easy to express analytically. A useful alternative parameterization is that of Q and natural frequency of pole position (QNF parameterization), with $\omega_n^2 = \omega_r^2 + b^2$ and $Q = b/(2\omega_n)$, in which variation of the nondimensional Q or damping factor $1/(2Q)$ models a physical change of damping.

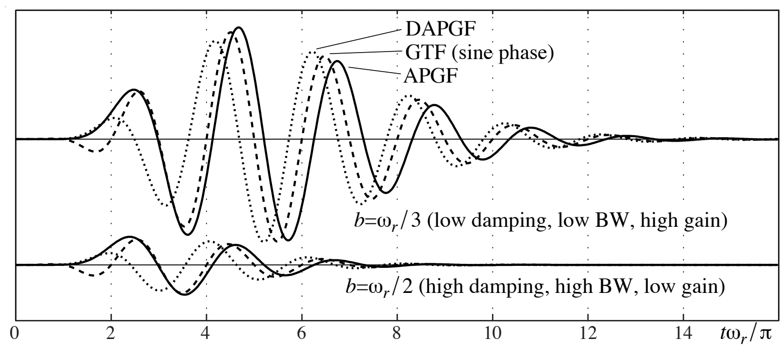

Figure 2: Impulse responses of the APGF, DAPGF, and sine-phase GTF from Figure 1. Note that the GTF's zero crossings are equally spaced in time, while those of the APGF and DAPGF are stretched out in the early cycles.

The magnitude of the GTF transfer function is almost exactly symmetric about a CF of ω_r. In contrast, the APGF and DAPGF are asymmetric (more in line with auditory reality) and have a CF somewhat below ω_r. At low frequencies, the APGF gain is nearly constant (near unity), so we say that it has a *flat* low-frequency tail. The gain of the high-frequency tail, by contrast, falls very rapidly with frequency. The DAPGF, due to the differentiator, has zero gain at DC and a slope of 6 dB per octave at low frequencies, so we say that it has a *sloping* low-frequency tail. Its passband and high-frequency tail are much like the APGF's, although the DAPGF is less asymmetric.

The APGF and DAPGF variants of the GTF thus have the advantage that their transfer functions can have a parameter-dependent passband gain, to implement a level-dependent nonlinearity, while maintaining a fixed tail gain, or linear low-frequency response. This property is useful in modeling psychophysical and physiological data.

3 Filter Cascades

A cascade filterbank is a natural approximate model, via the Liouville–Green (or WKB) method, of the distributed filtering in a nonuniform wave-propagation medium such as the cochlea.[3] Local complex wavenumber solutions allow us to

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design filters to match the gain and phase of a small section of the cochlea for forward-propagating waves of any frequency. A cascade of such filters can simultaneously compute the filtered model output at a large number of places, or taps, corresponding to a set of characteristic frequencies in a filterbank.

If each filter in such a cascade filterbank is an all-pole filter, then, at every tap, the composite transfer function from the input through all the stages up to that tap is an all-pole transfer function.

The simplest all-pole filter stage that leads to a pseudoresonant response is a two-pole filter, and the resulting APFC is very closely related to the APGF. The differences are that the poles are not coincident, and that each successive tap response has a higher order. The APFC's transfer functions are not quite as sharp and asymmetric as those of the APGF, due to the pole positions being distributed. Figure 3 compares typical APFC (one tap of a filter cascade) and APGF transfer-function magnitudes.

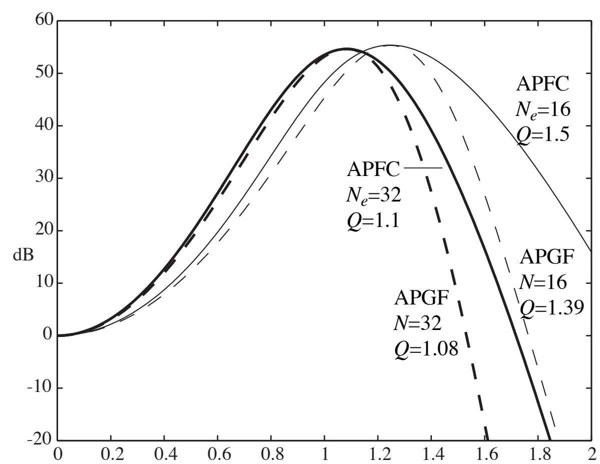

Figure 3: Comparison of APGF and APFC transfer functions. The frequency axis is labeled in units of the tap natural frequency, and parameters are noted in term of pole Q. The APFC needs a large number of stages per factor of e frequency change (N_e) to approximate the APGF shape.

4 Asymmetric Auditory Filters

The APGF is similar in result to Irino's *gammachirp* filter (GCF), which was derived from time–scale optimality but inspired by the need for an asymmetric variant of the GTF.[4] The GCF adds a free parameter to the GTF, controlling how much of a log-time term is added to the phase of the tone:

$$g(t) = t^{N-1} \exp(-bt) \cos(\omega_r t + \phi + c \ln t) \tag{5}$$

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The result is a controllable asymmetry that can be zero or of either sign. The APGF, DAPGF, and APFC, on the other hand, have an inherent asymmetry that depends on the pole positions, but have no additional parameters. For auditory filters, the degree of asymmetry seems to be reasonable; we have not done careful fitting yet. Of course, with the extra degree of freedom, the GCF always will be able to provide a closer fit to asymmetry data than can the simpler all-pole models.

The log-time term (with negative c, for the proper sign of asymmetry) corresponds to an instantaneous frequency that starts low and rises hyperbolically to approach the ringing frequency. A similar chirp effect has been pointed out in physiological data,[5] and can also be seen in the instantaneous frequency calculated from an APGF impulse response.

The GCF does not have the linear low-frequency tail, as parameters are varied, that characterizes the all-pole filters—it inherits the GTF's tail sensitivity to parameters (whether the GCF has a rational transfer function is not presently clear, so we cannot claim that it inherits the GTF's zeros).

5 Wide-Dynamic-Range Compression

One of the most important nonlinear functions of the cochlea is its compression of a wide range of sound intensities into a narrower range of cochlear motion intensities at the sensor array, for frequencies near CF (while maintaining near linearity for lower frequencies, and not enough gain to measure at higher frequencies). Studies of cochlear mechanical response since about 1970 have repeatedly demonstrated this frequency-dependent compression in live cochleas, and its absence in dead cochleas.[6]

In live cochleas, the overall input–output intensity curves for frequencies near CF have a slope of typically 0.25 to 0.5 on a log-log plot. The exact slope of this compression nonlinearity depends on the preparation, on the frequency and intensity range to a lesser extent, and on whether the response is measured at a fixed frequency or at the frequency of greatest response, which shifts slightly with level.

Since a wave picks up energy as it travels across a range of places, each little increment of place needs to contribute only a small amount of gain. If the filter cascade model has stages that model small place increments, each filter will need to contribute only a small gain; as the overall gain changes, each filter will have to change only slightly. Similarly, each pole in an APGF will need only a small Q change to effect a large overall gain change.

There are two general forms of compressive nonlinearity that are important to consider, and it is likely that both operate in the real cochlea: instantaneous nonlinear distortion, and feedback parameter variation. For example, the nonlinear model of Kim and associates is actually an APFC with a compressive nonlinearity in each stage;[7] Kim's later suggestions incorporate parameter feedback.[8]

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6 Conclusions

Removing the zeros from the popular GTF to make the APGF improves its applicability to auditory modeling in two ways: by decoupling a linear low-frequency tail from the variable-gain peak, and by adding a moderate asymmetry correlated with realistic group-delay dispersion or chirping.

The APFC is an all-pole model, related to, but slightly generalized from, the APGF, that uses noncoincident poles to two advantages: efficient realization of a whole bank of filters in one structure, and an analytic link to underlying wave-propagation models of cochlear filtering.

These all-pole structures provide a natural coupling among gain, bandwidth, and CF shift as we vary one parameter to model a level-dependent nonlinearity in the cochlea.

Acknowledgments

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References

1. Flanagan, J.L. (1960). Models for approximating basilar membrane displacement, *Bell Sys. Tech. J.* **39** 1163–1191.
2. Slaney, M. (1993). An efficient implementation of the Patterson-Holdsworth auditory filter bank, Apple Computer TR #35, Cupertino, CA.
3. Lyon, R.F. and Mead, C.A. (1988) An analog electronic Cochlea, *IEEE Trans. ASSP* **36** 1119–1134.
4. Irino, T. (1996). A 'gammachirp' function as an optimal auditory filter with the Mellin transform, *IEEE ICASSP-96,* Atlanta, 981–984.
5. Møller, A.R. and Nilsson, H.G. (1979) Inner ear impulse response and basilar membrane modelling, *Acustica* **41** 258–262.
6. Rhode, W.S. (1971) Observations of the vibration of the basilar membrane in squirrel monkeys using the Mössbauer technique, *J. Acoust. Soc. Am.* **49** 1218–1231.
7. Kim, D.O., Molnar, C.E., Pfeiffer, R.R. (1973) A system of nonlinear differential equations modeling basilar-membrane motion, *J. Acoust. Soc. Am.* **54** 1517–1529.
8. Kim, D.O. (1986) Active and nonlinear biomechanics and the role of outer-hair-cell subsystem in the mammalian auditory system, *Hear. Res.* **22** 105–114.