Photographic Technology

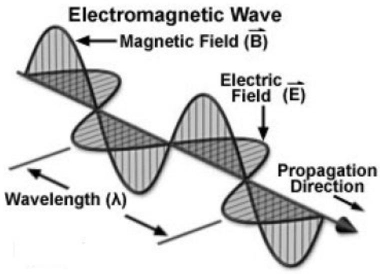
Lecture 8: March, 14 2007
Diffraction, gratings and spectroscopy

Rom Clement

Diffraction

What's light?

- Maxwell equation
  - Coupling between magnetic field H [A/m] and electric field E [V/m] $\vec{E}(\vec{r},t)=\Re\{\vec{E_0}(\vec{r})e^{-i\omega t}\}$ $\vec{H}(\vec{r},t)=\Re\{\vec{H_0}(\vec{r})e^{-i\omega t}\}$
  - Intensity of light is time average of Poynting vector
    $\langle\Pi\rangle=\frac{c}{8\pi}\Re\left(\vec{E_0}\times\vec{H_0}^*\right)$ $I=\frac{c}{4\pi}\sqrt{\frac{\varepsilon}{\mu}}\langle\vec{E}^2\rangle=\frac{c}{4\pi}\sqrt{\frac{\varepsilon}{\mu}}\langle\vec{H}^2\rangle$
  - For a plane wave
    - Intensity of light proportional to E^2
  - Contrast $C=\frac{I_{\max}-I_{\min}}{I_{\max}+I_{\min}}$



Assumptions

- Vacuum
  (simplified Maxwell's equations)

$$div(\vec{B})=0$$
$$div(\vec{E})=0$$
$$\overrightarrow{rot}(\vec{E})=-\frac{\partial\vec{B}}{\partial t}$$
$$\overrightarrow{rot}(\vec{B})=\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

- Media
  - Linear
  - Homogenous
  - Isotropic
- Plane wave
  - Far enough from the source so that ~ plane wave

Young's experiment

$$\vec{E} = \vec{E_1} + \vec{E_2}$$

$$\vec{E}^2 = \vec{E_1}^2 + \vec{E_2}^2 + 2\vec{E_1}\cdot\vec{E_2} = I_1 + I_2 + J_{12}$$

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\delta) \rightarrow I_{max} = I_1 + I_2 + 2\sqrt{I_1 I_2}, \quad I_{min} = I_1 + I_2 - 2\sqrt{I_1 I_2}$$

$$\delta = \frac{2\pi}{\lambda}\frac{nxd}{a}$$ n integer

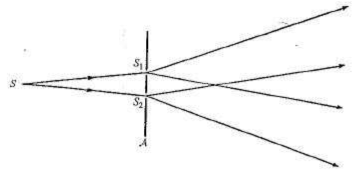
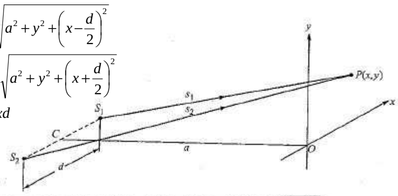


Fig. 7.2 Young's experiment.

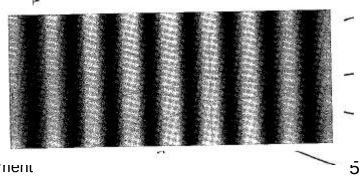
$$s_1 = S_1P = \sqrt{a^2 + y^2 + \left(x - \frac{d}{2}\right)^2}$$

$$s_2 = S_2P = \sqrt{a^2 + y^2 + \left(x + \frac{d}{2}\right)^2}$$

$$s_2^2 - s_1^2 = 2xd$$

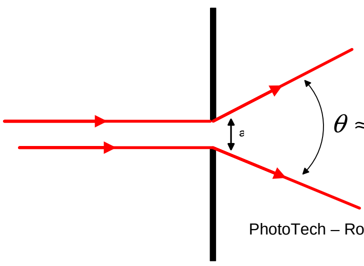


- Coherence Criteria
  - Same wavelength
  - Same source
  - Max difference of optical path (coherence's length of the source)



What's diffraction

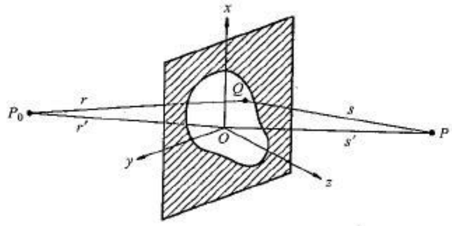
- Diffraction occurs when the wavelength of a wave (visible light, IR, UV, sound, radio waves…) is about the same size as an object
  - Example1: a needle's hole in a sheet of paper generates diffraction for light
  - Example2: an opened door can represent a hole in a wall for which diffraction will occur



$$\theta \approx \frac{\lambda}{a}$$ a ~ 0.1mm

Fraunhofer Diffraction

- Aperture A in a sheet
  - Po is source creates a "plane wave"
  - P(p,q) is point on a screen located at (p,q)
  - U amplitude of the wavefront
  - (ξ,η) coordinates of a point Q in the aperture
  - $G(\xi,\eta)$ pupil function (depends on transmission characteristics)



General diffracted amplitude

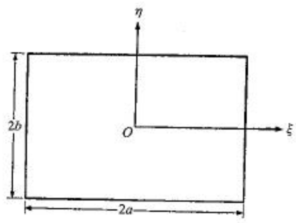
$$U(p,q)=\iint G(\xi,\eta)e^{-\frac{2i\pi}{\lambda}(p\xi+q\eta)}d\xi d\eta$$

$G(\xi,\eta)=\text{constant}$ at point of the opening

$G(\xi,\eta)=0$ at points outside the opening

In our case we will consider
$$\begin{cases} G(\xi,\eta)=1 & \text{at point of the opening} \\ G(\xi,\eta)=0 & \text{at points outside the opening} \end{cases}$$

Diffraction of a rectangular aperture

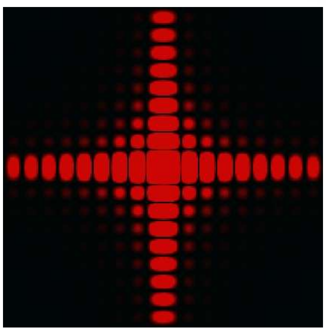


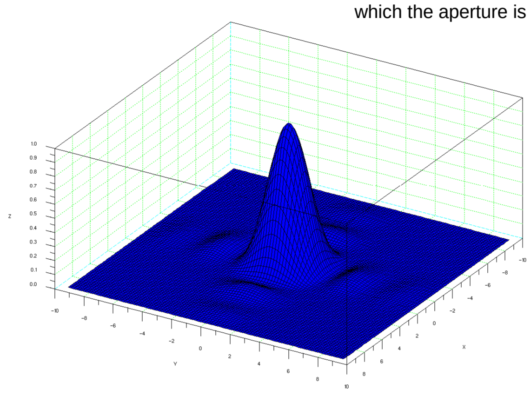
Diffraction of rectangular aperture: 2a width, 2b height

$$U(P)=C\int_{-a}^{a}\int_{-b}^{b}e^{-ik(p\xi+q\eta)}d\xi d\eta=C\int_{-a}^{a}e^{-ik(p\xi+q\eta)}d\xi\int_{-b}^{b}e^{-ik(p\xi+q\eta)}d\eta$$

$$I(P)=|U(P)|^2=\left(\frac{\sin(kpa)}{kpa}\right)^2\left(\frac{\sin(kqb)}{kqb}\right)^2 I_0 \qquad k=\frac{2\pi}{\lambda}$$

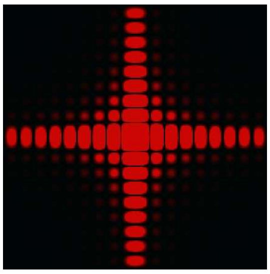
p,q direction (angle) from which the aperture is seen





2D plots, Rectangular and Circular Aperture

- Rectangular Aperture

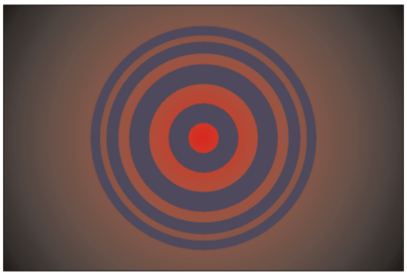


$$I(P) = |U(P)|^2 = \left(\frac{\sin(kpa)}{kpa}\right)^2 \left(\frac{\sin(kqb)}{kqb}\right)^2 I_0$$

$$k = \frac{2\pi}{\lambda}$$

p,q angle from which the aperture is seen

- Circular aperture

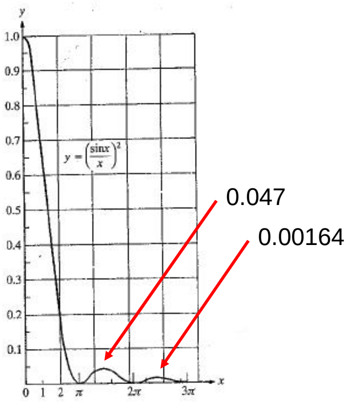


$$I(P) = |U(P)|^2 = \left[\frac{2J_1(kaw)}{kaw}\right]^2 I_0$$

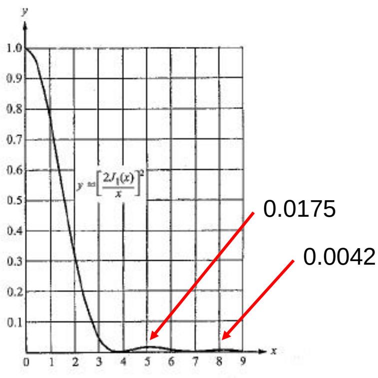
w direction angle from which the aperture is seen

Comparison rectangular/circular aperture

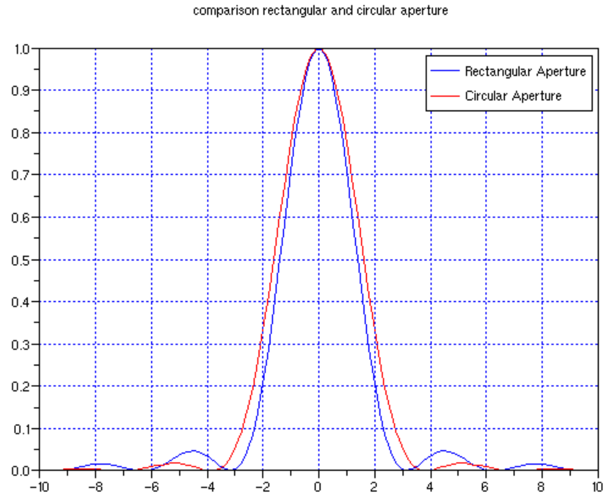
- Rectangular opening



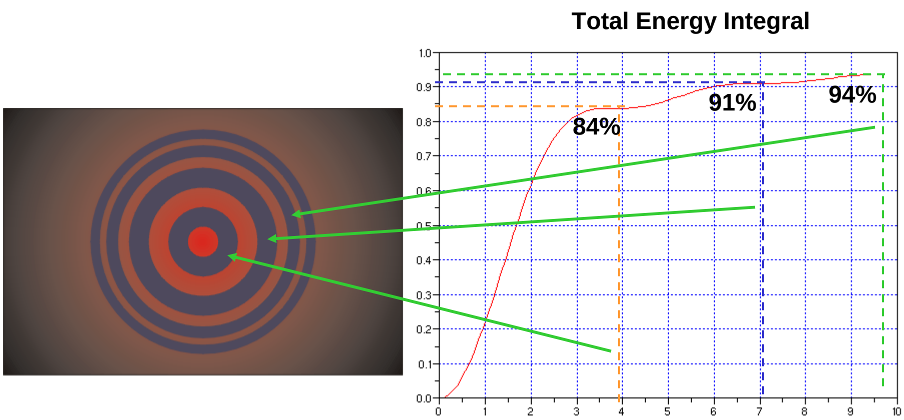
- Circular opening



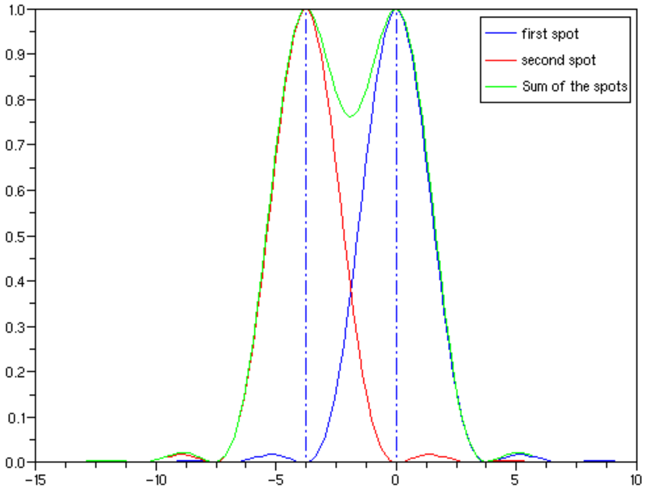
Comparison rectangular/circular aperture



How to define the “spot”?



The problem of Resolution



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How to determine when 2 objects are distinguishable?

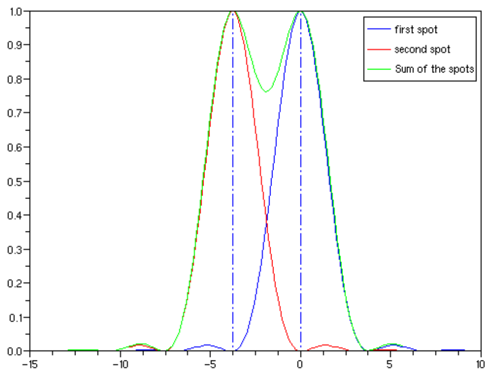
- Abbe's criterion (1873)
  - When maximum of first PSF coincides with 1st minimum of 2nd PSF

$$\theta_{Abbe} = 1.2196\frac{\lambda}{D}$$

- Rayleigh's criterion (1879)
  - Airy disk contains 84% of energy
  - 2 points can be resolved if centers of Airy disks are separated by ½ the diameter of the Airy disk

$$\theta_{Rayleigh} = 1.22\frac{\lambda}{D}$$

Abbe's/Rayleigh Criterion



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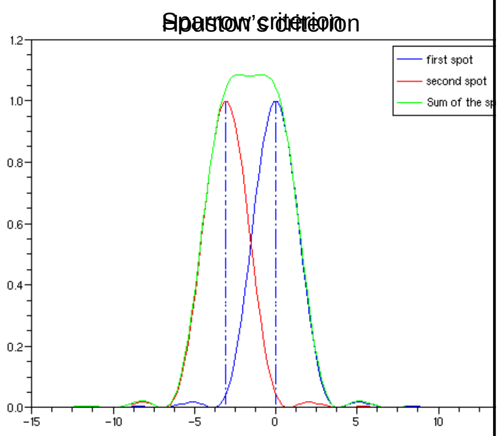
How to determine when 2 objects are distinguishable?

- Sparrow's criterion (1916)
  - Based on summed line profiles
  - Where the saddle point first develops

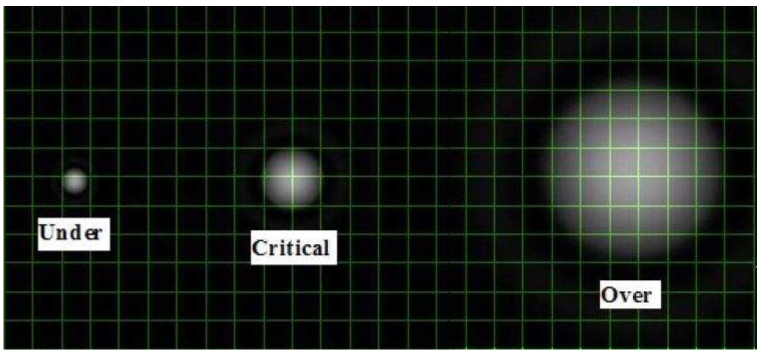
$$\theta_{Sparrow} = 0.94\frac{\lambda}{D}$$

- Houston's criterion (1926)
  - FWHM (Full Width at Half Maximum)
  - The two diffraction patterns have to be separated by the FWHM

- Sparrow's is the least stringent, then Houston's then Rayleigh's ~= Abbe

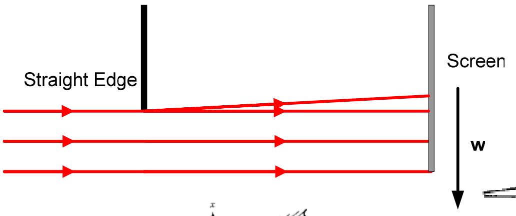


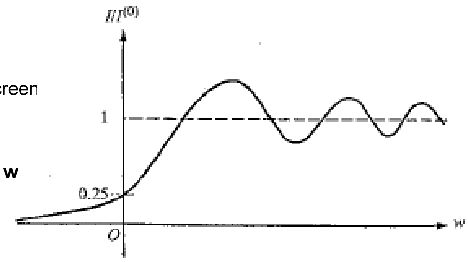
Which pixel's size to chose?

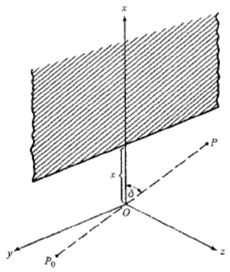


- The optimum: Nyquist criteria
  - i.e. 2 pixels inside the airy disk

Another example: diffraction of an straight edge







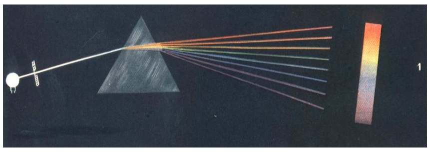
$$C(\infty)+iS(\infty)=\int e^{i(\pi/2)\tau^2}d\tau$$

$$I=\frac{1}{2}\left\{\left[\frac{1}{2}+C(w)\right]^2+\left[\frac{1}{2}+S(w)\right]^2\right\}I^0$$

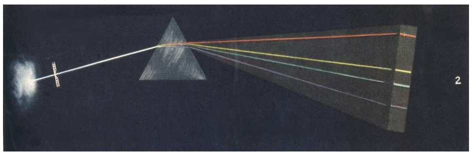
Gratings and Spectroscopy

Spectroscopy's Principles

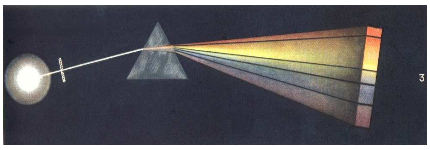
Continous spectrum



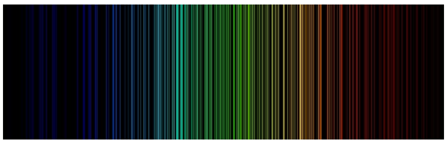
Emission spectrum



Absorption spectrum



Spectrum



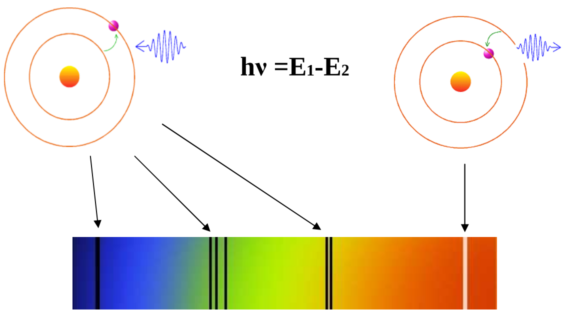
In order to have a spectrum, we need to scatter the light, prism is a way but not the only one…

Line emission: theory

Absorption

$h\nu = E_1 - E_2$

Emission



Ritz's Law

$1/\lambda = R_H (1/n^2 - 1/m^2)$

R_H Rydberg's constant 109 677 1/m
n,m integers and m>n

Lyman's serie (n=1), Balmer (n=2), Paschen (n=3)

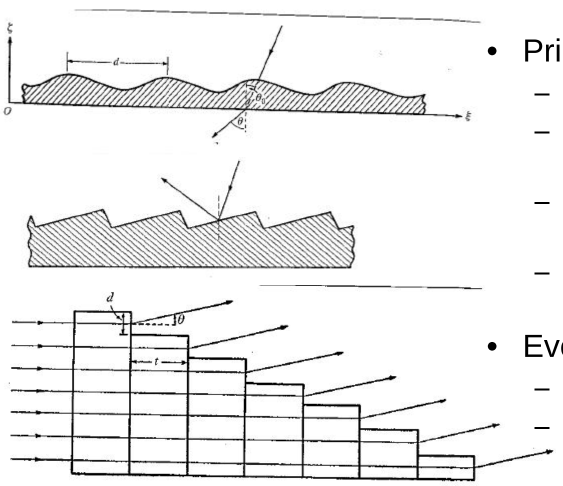
Balmer

m=3 Hα (λ=6563 Å)
m=4 Hβ (λ=4861 Å)
m=5 Hγ (λ=4340 Å)
m=6 Hδ (λ=4101 Å)….

Lines

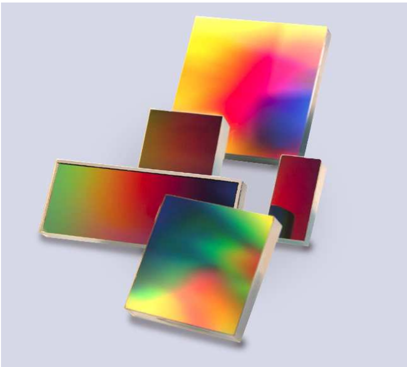
- The wavelength of the lines allow to characterize precisely the components of the light source (or absorbing source) but the study of the distortion of these lines allows to learn more…
- Line spreading
  - Natural width (2/1000 Å)
  - Thermal shaking (Doppler-Fizeau's effect) (1/100 Å)
  - Spreading due to high pressure
  - Stark effect, electric field (several Angströms)
  - Star rotating speed (Doppler-Fizeau effect) (several Angströms)
- Lines split
  - Zeeman effect (triple lines, 1/400 Å), magnetic field

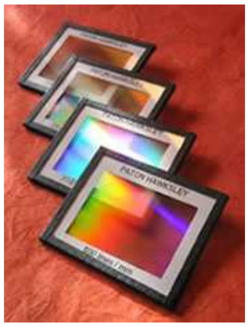
Gratings



- Principles
  - Repetition of a pattern…
  - …introduces a slight difference of optical path…
  - …which introduces a difference of phase…
  - … and generates interferences
- Everyday life's examples
  - CD's/DVD iridescence
  - Credit cards' hologram…

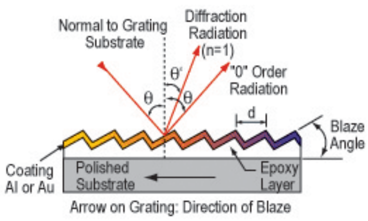
Reflective and transmission: gratings

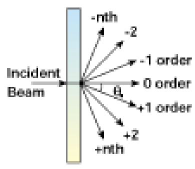




Gratings: examples

- Reflective gratings
  - Stiffer
    - Aluminized epoxy layer glued on a thick glass
  - Higher efficiency (blazed)
  - 2,500 lines/mm
  - Price
    - $130 for 30mm*30mm (visible light)
    - $250 for 50mm*50mm (visible light)
- Transmission gratings
  - Polyester film
  - 1,000 lines/mm
  - Lower efficiency
  - Cheaper $25





Transmission gratings, the math

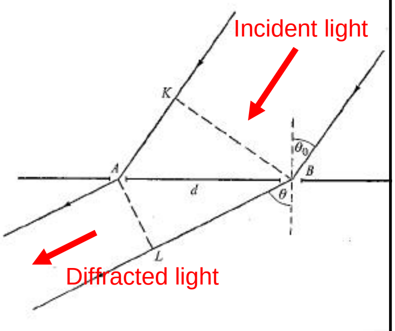
Assume a transmission grating made of N slits (2D case) of aperture's size 's' and distant of distance 'd

The difference of optical path between 2 successive beams is $\delta = \frac{2\pi}{\lambda} d(\sin(\theta) - \sin(\theta_0))$

If we sum the amplitude over all N slits we obtain

$$U(p) = U^0(p)\sum_{n=0}^{N-1} e^{ikndp} = U^0(p)\frac{1-e^{-iNkdp}}{1-e^{-ikdp}}$$

$$I(P) = |U(P)|^2 = \frac{1-\cos(Nkdp)}{1-\cos(kdp)} I^0(p) = \left[\frac{\sin(Nkdp/2)}{\sin(kdp/2)}\right]^2 I^0$$



where $I^0(p) = |U^0(p)|^2$ and $U^0(p)$ diffracted amplitude of a single slit

$$I(P) = \left[\frac{\sin(Nkdp/2)}{\sin(kdp/2)}\right]^2 \operatorname{sinc}^2(ksp/2)$$

$$k = \frac{2\pi}{\lambda}$$

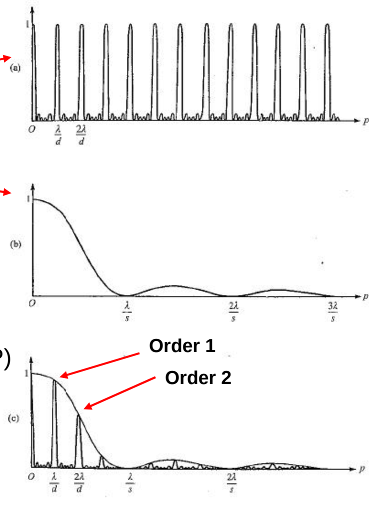
$$p = \sin(\theta) - \sin(\theta_0)$$

Transmission gratings, the maths

- Normalized intensity

$$\frac{1}{N^2} I(P) = \left[\frac{\sin(Nkdp/2)}{N\sin(kdp/2)}\right]^2 \operatorname{sinc}^2(ksp/2)$$

Modulation function due to N slits — Intensity of one single slit



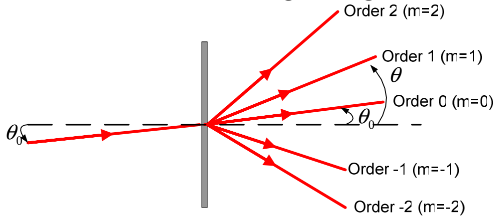
For a given wavelength, pics appear at specific angular positions

Resolving Power $\frac{\lambda}{\Delta\lambda} = \frac{Nd|\sin(\theta) - \sin(\theta_0)|}{\lambda}$

Angular dispersion $\frac{\partial\theta}{\partial\lambda} = \frac{1}{\cos(\theta)} \frac{\sin(\theta) - \sin(\theta_0)}{\lambda}$

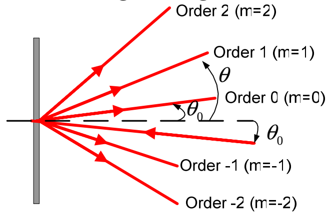
Overlapping of orders

- Transmission grating



$$\sin(\theta) - \sin(\theta_0) = \frac{m\lambda}{d} \qquad (m = 0, \pm 1, \pm 2, ...)$$

- Reflective grating



$$\sin(\theta) + \sin(\theta_0) = \frac{m\lambda}{d}$$

m is the order of interference

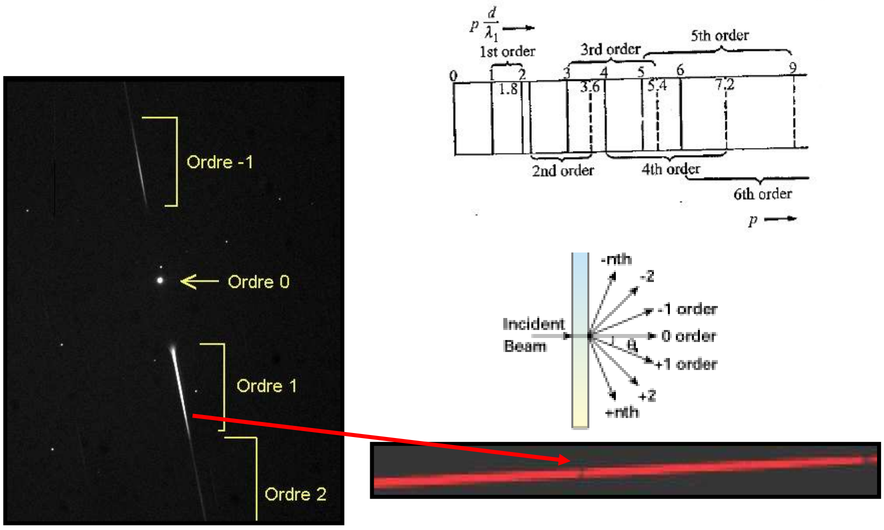
d is the distance between 2 groves (or lines, or step…)

But $\lambda_{visible} \in [400\,nm, 800\,nm]$

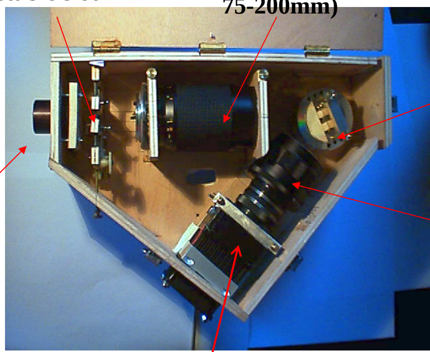
Assume 100 groves/mm, i.e. d = 10µm and θ_0=10°

	$\lambda_1 = 400nm$	$\lambda_2 = 800nm$
m=1	12.3°	14.7°
m=2	14.7°	19.5°
m=3	17°	Pb !! 24.4°

Overlapping of grating spectra



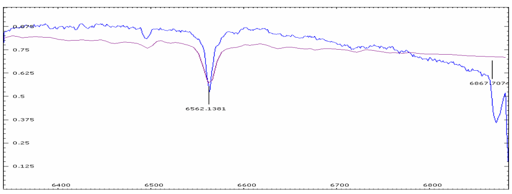
Amateur spectroscope



Post processing steps (Procyon)

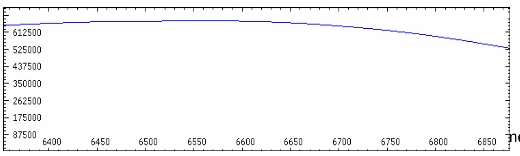
- Crude image (~ 6700Å)

- Alignment
- Binning



- Spectrum Normalization
- Comparison with reference spectrum
- Polynom of interpolation

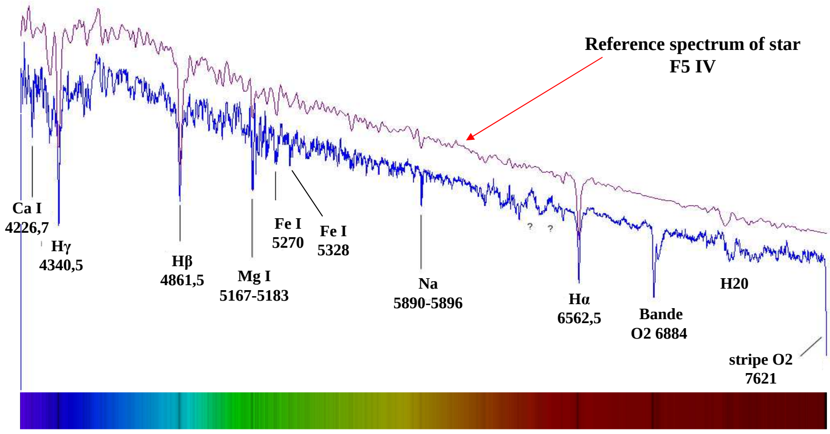
$$\lambda = 2{,}63x10^{-5}P(i)^2 + 0{,}825P(i) + cste$$



- Instrument response

Entire spectrum of Procyon (type F5 IV)

Stiching of 9 calibrations



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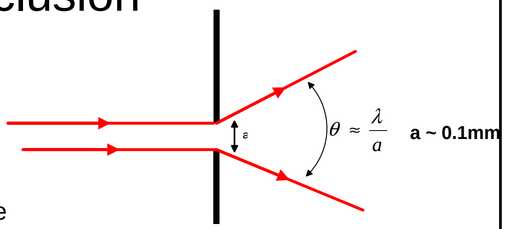
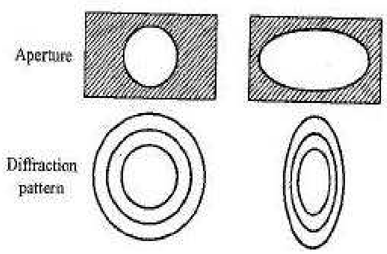
Conclusion

- Diffraction’s phenomena comes from the <u>wave</u> approach of light
  - A perfect point source is always diffracted by an opening
  - Its diffracted pattern is called the airy disk
  - Diffraction usually size the aperture of optical components
  - Diffraction size the optimum sensor spatial resolution
- Spectroscopy
  - How diffraction can be used to analyze light: gratings, a pattern of 0.1µm allows to extract information at the Angström’s scale = 1^{e-10}m (factor of 10^3!)





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References

- Books
  - Principles of Optics, 7th edition, Max Born and Emil Wolf (hardcore but very very good!)
- Websites
  - Wolfram: http://scienceworld.wolfram.com/physics/
  - Definition of focal length: http://en.wikipedia.org/wiki/Lens_%28optics%29
  - Edmund Optics for gratings
- Astronomical spectroscopy for amateur
  - http://astrosurf.com/buil/
    - 6 different spectrographs for different budgets and objectives
    - Visual Spec: spectrum analyzer software
    - IRIS: Astronomical images processing software
    - All for free!

Appendixes

Diffraction of a circular aperture: the maths

Change for Cartesian reference to cylindrical reference

$$\rho\cos(\theta) = \xi w\cos(\psi) = p$$
$$\rho\sin(\theta) = \xi w\sin(\psi) = q$$

Leads to the following integral $U(P) = C\int_0^a \int_0^{2\pi} e^{-ik\rho w\cos(\theta-\psi)} \rho d\rho d\theta$

Bessel function

$$\frac{i^{-n}}{2\pi}\int_0^{2\pi} e^{ix\cos(\alpha)} e^{in\alpha} d\alpha = J_n(x)$$

and recurrence relation $\frac{d}{dx}\left[x^{n+1}J_{n+1}(x)\right] = x^{n+1}J_n(x)$

So the integral can be expressed as a function of J_0

$U(P) = 2\pi C\int_0^a J_0(k\rho w)\rho d\rho$ Or $U(P) = CD\left[\frac{2J_1(kaw)}{kaw}\right]$ with $D = \pi a^2$

And ultimately we obtain the intensity's expression:

$$I(P) = |U(P)|^2 = \left[\frac{2J_1(kaw)}{kaw}\right]^2 I_0$$

a aperture radius [m]
k wave vector [1/m]
w angle from which the aperture is seen

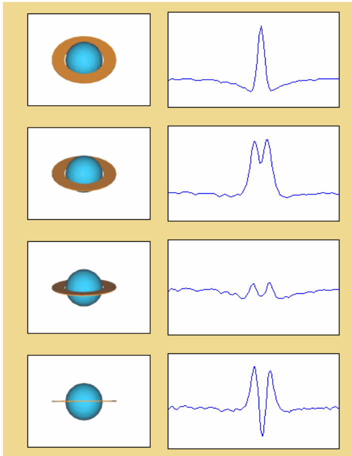
Stars' Classification

Type	Color	External layer's temperature	Principle characteristics	Examples
O	Blue	> 25000 K	Ionized helium Fort strong UV radiation	10 Lacerta
B	Blue	11 000 à 25 000 K	Helium absorption lines	Rigel Spica
A	Blue	7500 à 11 000 K	Hydrogen emission lines	Sirius Vega
F	Blue to white	6000 à 7500 K	Metals lines	Canopus Procyon
G	White to yellow	5000 à 6000 K	Absorption lines of metal atoms	Soleil Capella
K	Orange to red	3500 à 5000 K	Several metals lines	Arcturus Aldebaran
M	Red	< 3500 K	Stripes of Titanum oxydes molecules	Betelgueuse Antares

There is a much, much more well defined classification inside each category

Type of information extracted from light

- Analysis of lines
  - Chemical composition of the star
- Analysis of lines width/distortion
  - Characterization of emission milieu: pressure, magnetic field (Zeeman effect)…
  - Doppler's effect :
    - Extraction of star's rotational velocity
    - Extraction of expansion speed of star's envelope



➡ Need high resolution of spectroscopes